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Displacement and curvature effects in a wall jet

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This paper presents a solution of the second-order boundary-layer equations for the two-dimensional case of a wall jet on a curved surface. The outer flow is obtained by means of a conformal transformation, and general solutions for the displacement and curvature effects are given both as series and as integrals. These solutions are applied to symmetrical flow over a parabolic surface, the wall jet being either outside or inside.

1. Introduction

The second-order equations for viscous flow at high Reynolds number were obtained by Van Dyke (1962) from the method of matched asymptotic expansions. So far there have been few full solutions of these equations, and these have been surveyed by Van Dyke (1969). The study reported here was undertaken in order to provide a complete solution of the equations in a specific case and to compare the importance of the displacement and curvature effects.

The problem chosen for this purpose is that of the two-dimensional flow produced by a wall jet on a curved surface. The advantages of this choice are first, that the outer flow is at rest to first order so that the second-order flow may be obtained by potential theory and in particular by conformal transformation; second, that the first-order boundary layer has an analytical solution, perturbations to which satisfy equations that can be reduced to hypergeometric form.

The wall jet in this problem is supposed to be produced by blowing tangentially along the surface in opposite directions from narrow slits. The boundary layers thus formed draw in fluid from their surroundings and this entrainment drives the second-order outer flow. This outer flow and the curvature of the surface cause the second-order perturbations to the boundary layer that are described by the title of this paper.

As introduced by Glauert (1956), the wall jet is a steady flow which satisfies the first-order boundary-layer equations for an incompressible fluid, namely

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{s}}+\bar{v}\frac{\partial \bar{u}}{\partial \bar{n}} = \nu\frac{\partial^2 \bar{u}}{\partial \bar{n}^2}, \tag{1}$$

$$\frac{\partial \bar{u}}{\partial \bar{s}}+\frac{\partial \bar{v}}{\partial \bar{n}} = 0, \tag{2}$$

with the conditions

$$\bar{u} = \bar{v} = 0 \quad \text{at} \quad \bar{n} = 0, \quad \bar{u}\to 0 \quad \text{as} \quad \bar{n}\to\infty. \tag{3}$$

Here $\bar{s}, \bar{n}$ are curvilinear co-ordinates along and at right angles to a fixed surface, $\bar{u}, \bar{v}$ are the corresponding velocity components, and ν is the kinematic viscosity of the fluid.

This flow is characterized by the quantity

$$F = \int_0^\infty \bar{u}^2 \bar{\psi}\, d\bar{n}, \tag{4}$$

where $\bar{\psi}$ is the stream function defined so that

$$\bar{u} = \partial\bar{\psi}/\partial\bar{n}, \quad \bar{v} = -\partial\bar{\psi}/\partial\bar{s}, \quad \bar{\psi}(\bar{s}, 0) = 0. \tag{5}$$

The constancy of F follows from the equation

$$\frac{\partial}{\partial\bar{s}}(\bar{u}^2\bar{\psi}) + \frac{\partial}{\partial\bar{n}}\left(\bar{u}\bar{v}\bar{\psi} - \nu\bar{\psi}\frac{\partial\bar{u}}{\partial\bar{n}} + \tfrac{1}{2}\nu\bar{u}^2\right) = 0, \tag{6}$$

which is a consequence of (1), (2) and (5).

Glauert obtained a similarity solution of the equations in the form

$$\bar{\psi} = (40\nu F\bar{s})^{\frac{1}{4}} f(\eta), \tag{7}$$

$$\eta = \tfrac{1}{4}(40F/\nu^3\bar{s}^3)^{\frac{1}{4}}\,\bar{n}, \tag{8}$$

where $f(\eta)$ satisfies the equation

$$f''' + ff'' + 2f'^2 = 0, \tag{9}$$

with

$$f(0) = f'(0) = f'(\infty) = 0. \tag{10}$$

The numbers in (7) and (8) have been chosen so that the appropriate solution of (9) has $f(\infty) = 1$, and then

$$f(\eta) = g^2(\eta) \tag{11}$$

where

$$g' = \tfrac{1}{3}(1-g^3), \tag{12}$$

from which

$$\eta = \tfrac{1}{2}\log\left\{\frac{1+g+g^2}{(1-g)^2}\right\} + \sqrt{3}\tan^{-1}\left(\frac{g\sqrt{3}}{g+2}\right). \tag{13}$$

In the later parts of this paper, integrals involving f or its derivatives may often be evaluated by taking Glauert's variable g as the variable of integration.

The general equations for the first- and second-order terms in both the outer and inner flows are given by Van Dyke (1962, 1969), together with the necessary matching conditions. These terms represent successive approximations in the limit $R \to \infty$, where R is a Reynolds number for the flow. In the case of the wall jet we introduce a length scale l for the surface over which the jet flows, and the characteristic velocity in the boundary layer is then

$$U_c = (40F/\nu l)^{\frac{1}{4}}, \tag{14}$$

which leads to the Reynolds number

$$R = (40Fl/\nu^3)^{\frac{1}{4}}. \tag{15}$$

For the outer flow we use Cartesian co-ordinates $(\bar{x}, \bar{y}) = (lx, ly)$. There is no first-order flow and we can write the second-order velocity as $R^{-\frac{1}{2}}U_c\mathbf{V}$ and take the pressure as $R^{-1}\rho U_c^2 P$. Then $\mathbf{V}, P$ satisfy the ordinary inviscid equations

$$(\mathbf{V}.\nabla)\mathbf{V} = -\operatorname{grad} P, \quad \operatorname{div}\mathbf{V} = 0. \tag{16}$$

Since the flow is at rest at infinity we can introduce a velocity potential $R^{-\frac{1}{4}}U_c l\Phi$ and a stream function $R^{-\frac{1}{4}}U_c l\Psi$ such that

$$\mathbf{V} = \operatorname{grad}\Phi = \operatorname{curl}(\Psi\mathbf{k}), \quad P = P_\infty - \tfrac{1}{2}\mathbf{V}^2. \tag{17}$$

In the boundary layer the variables are made dimensionless by writing

$$\bar{s} = ls, \quad \bar{n} = R^{-\frac{1}{2}}ln, \quad \bar{u} = U_c u, \quad \bar{v} = R^{-\frac{1}{4}}U_c v, \quad \bar{\psi} = R^{-\frac{1}{4}}U_c l\psi. \tag{18}$$

The pressure is taken as $\rho U_c^2 p$. The dependent variables are expanded in powers of $R^{-\frac{1}{4}}$, so that

$$u = u_1 + R^{-\frac{1}{4}}u_2 + \ldots, \text{etc.} \tag{19}$$

and Van Dyke's equations are then obtained as the coefficients of the various powers of R in the Navier–Stokes equations with s, n as independent variables.

The first-order velocity components (u_1, v_1) satisfy the dimensionless forms of (1) and (2), and the first-order pressure p_1 is constant. The matching conditions are

$$\lim_{n\to\infty} u_1 = 0, \quad V_n(s) = \lim_{n\to\infty}(v_1 - n\,\partial v_1/\partial n), \tag{20}$$

where $V_n(s)$ is the component of $\mathbf{V}$ normal to the surface, evaluated at the surface. Glauert's solution takes the form

$$\left.\begin{aligned} \psi_1 &= s^{\frac{1}{4}}f(\eta), \quad \eta = \tfrac{1}{4}s^{-\frac{3}{4}}n, \\ u_1 &= \tfrac{1}{4}s^{-\frac{1}{2}}f'(\eta), \quad v_1 = -\tfrac{1}{4}s^{-\frac{3}{4}}(f - 3\eta f'). \end{aligned}\right\} \tag{21}$$

The second-order boundary-layer equations involve the curvature of the surface, which is written as $l^{-1}\kappa(s)$, where $\kappa(s) > 0$ if the surface is convex to the flow. These equations are

$$u_1\frac{\partial u_2}{\partial s} + u_2\frac{\partial u_1}{\partial s} + v_1\frac{\partial}{\partial n}(u_2 + \kappa n u_1) + v_2\frac{\partial u_1}{\partial n} = -\frac{\partial p_2}{\partial s} + \frac{\partial^2 u_2}{\partial n^2} + \kappa\frac{\partial}{\partial n}\left(n\frac{\partial u_1}{\partial n}\right), \tag{22}$$

$$\kappa u_1^2 = \frac{\partial p_2}{\partial n} + \kappa n\frac{\partial p_1}{\partial n}, \tag{23}$$

$$\frac{\partial u_2}{\partial s} + \frac{\partial}{\partial n}(v_2 + \kappa n v_1) = 0, \tag{24}$$

and are to be solved for u_2, v_2, p_2 subject to the no-slip conditions

$$u_2 = v_2 = 0 \quad \text{at} \quad n = 0 \tag{25}$$

and the matching conditions, which for the present problem are

$$u_2 \to V_t(s), \quad p_2 \to 0 \quad \text{as} \quad n \to \infty, \tag{26}$$

where $V_t(s)$ is the tangential component of the outer flow $\mathbf{V}$ at the surface of the body. The remainder of this paper is concerned with the solution of these equations in the case when the first-order flow is given by Glauert's similarity solution.

2. Outer flow

It will be assumed that the wall jet is two-sided and symmetrical, having the same strength F on each side. From (20) and (21) the boundary condition for the outer flow is then

$$V_n(s) = -\tfrac{1}{4}|s|^{-\frac{3}{4}}. \tag{27}$$

The surface is assumed to extend to infinity, in order to avoid the complication of boundary layers colliding on the far side of a closed body, and we need to determine the potential flow and in particular the function $V_t(s)$ in view of (26).

Since the region of flow extends to infinity it is convenient to derive it from a half plane by means of a conformal transformation. We write $z = x+iy$ for the physical plane, $Z = X+iY$ for the transformed plane, and assume that

$$z = F(Z) = \sum_0^\infty f_n Z^n \quad (|Z| < R^*), \tag{28}$$

where the region of flow is the image of $Y > 0$ and the points at infinity correspond. Without loss of generality we can suppose that the wall jet is at $z = 0$ and that this corresponds to $Z = 0$ so that $f_0 = 0$. The function $F(Z)$ is regular and $F'(Z) \neq 0$ in $Y \geqslant 0$, though there may be singularities in $Y < 0$. On the surface

$$ds = |dz| = |F'(X)|\, dX, \tag{29}$$

so that

$$s = S(X) = \int_0^X \{F'(X)\,\overline{F'(X)}\}^{\frac{1}{2}}\, dX \tag{30}$$

with s having the sign of X. If we consider the function

$$\bar{F}(Z) = \overline{F(\bar{Z})} = \sum_0^\infty \bar{f}_n Z^n \quad (|Z| < R^*), \tag{31}$$

we can extend the function $S(X)$ into the complex plane as

$$S(Z) = \int_0^Z \{F'(Z)\,\bar{F}'(Z)\}^{\frac{1}{2}}\, dZ. \tag{32}$$

Then $S(Z)$ is regular in $|Z| < R^*$ and is real for Z real. Near $Z = 0$, $S(Z) \sim |f_1| Z$. The singularities of $S(Z)$ in $Y > 0$ are where $\bar{F}(Z)$ is singular or $\bar{F}'(Z) = 0$.

The complex potential

$$w(Z) = \Phi(X, Y) + i\Psi(X, Y) \tag{33}$$

is regular in $Y > 0$ and satisfies

$$\Psi(X, 0) = \operatorname{sgn}(X)\, |S(X)|^{\frac{1}{4}} \tag{34}$$

with

$$dw/dZ \to 0 \quad \text{as} \quad |Z| \to \infty \quad \text{in} \quad Y > 0. \tag{35}$$

The condition (34) is satisfied by the function

$$w_1(Z) = -(\sqrt{2}+1-i)\, S^{\frac{1}{4}}(Z). \tag{36}$$

Hence

$$w(Z) = w_1(Z) + w_2(Z), \tag{37}$$

where $w_2(X)$ is real for real X. It is convenient to express the tangential velocity as

$$V_t(s) = U_1(s) + U_2(s), \tag{38}$$

where

$$U_1(s) = \frac{d}{ds}\Phi_1(X, 0) = -\operatorname{sgn}(s)\,\frac{\sqrt{2}+1}{4}\,|s|^{-\frac{3}{4}}. \tag{39}$$

Then $U_1(s)$ is the displacement flow due to a wall jet on a plane surface and $U_2(s)$ is the additional flow caused by the curvature of the surface. The series solution of the second-order boundary-layer equations in § 3 requires $U_2(s)$ to be expressed as a power series for small s, but the actual values of $U_2(s)$ are needed by the integral method of § 4.

The flow in the Z plane may be produced by a sink distribution along the X axis, the sink strength in an element dX being

$$2\,d\Psi = \tfrac{1}{2}|S(X)|^{-\frac{3}{4}}S'(X)\,dX \tag{40}$$

when allowance is made for flow on both sides. Thus if $Y \neq 0$

$$\frac{dw}{dZ} = \frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{|S(X)|^{-\frac{3}{4}}S'(X_1)}{X_1 - Z}\,dX_1. \tag{41}$$

If the Z plane is cut along the negative real axis this result can be expressed in terms of contour integrals passing above (C_+) and below (C_-) the cut as

$$\frac{dw}{dZ} = \left(\frac{1+(\sqrt{2}+1)\,i}{8\pi}\int_{C_+} + \frac{1-(\sqrt{2}+1)\,i}{8\pi}\int_{C_-}\right)\frac{S^{-\frac{3}{4}}(Z_1)\,S'(Z_1)}{Z_1 - Z}\,dZ_1, \tag{42}$$

where Z lies above both contours. If C_+ is moved across the pole at $Z_1 = Z$, an addition is made of

$$\tfrac{1}{4}(\sqrt{2}+1-i)\,S^{-\frac{3}{4}}(Z)\,S'(Z) = -\,dw_1/dZ.$$

Hence for Z real, or $|Z|$ small, (42) gives dw_2/dZ with Z lying between the contours. For $|Z|$ small, we can expand in powers of Z to get

$$\frac{dw_2}{dZ} = \sum_1^{\infty} n a_n Z^{n-1}, \tag{43}$$

where, after integrating by parts,

$$a_n = \mathscr{R}\left\{\frac{1+(\sqrt{2}+1)\,i}{\pi}\int_{C_+}\frac{S^{\frac{1}{4}}(Z_1)}{Z_1^{n+1}}\,dZ_1\right\}. \tag{44}$$

Thus

$$U_2(s) = \frac{1}{S'(x)}\sum_1^{\infty} n a_n X^{n-1} = \sum_0^{\infty} d_n s^n \quad \text{(say)}, \tag{45}$$

where the coefficients d_n can be calculated by use of the power series for $S(X)$, since $s = S(X)$.

If $Z = X$ is real we have similarly

$$U_2(s) = \frac{1}{S'(X)}\,\mathscr{R}\left\{\frac{1+(\sqrt{2}+1)\,i}{\pi}\int_{C_+}\frac{S^{\frac{1}{4}}(Z_1)}{(Z_1 - X)^2}\,dZ_1\right\}. \tag{46}$$

If the surface is symmetrical about the normal at the position of the wall jet then $S(X)$ and $U_2(s)$ are odd functions and

$$\int_{C_-}\frac{S^{\frac{1}{4}}(Z_2)}{(Z_2 - X)^2}\,dZ_2 = -\int_{C_+}\frac{e^{-\frac{1}{4}\pi i}\,S^{\frac{1}{4}}(Z_1)}{(Z_1 + X)^2}\,dZ_1,$$

so that

$$U_2(s) = \frac{2}{\pi}\,(1+(\sqrt{2}+1)\,i)\int_{C_+}\frac{XZ_1}{(Z_1^2 - X^2)^2}\,S^{\frac{1}{4}}(Z_1)\,dZ_1. \tag{47}$$

Similarly, if C_+ and C_- are divided symmetrically into L_+, R_+ and L_-, R_- we have

$$\int_{L_+}\frac{Z_1}{(Z_1^2 - X^2)^2}\,S^{\frac{1}{4}}(Z_1)\,dZ_1 = \int_{R_-}\frac{-\,e^{\frac{1}{4}\pi i}\,Z_2}{(Z_2^2 - X^2)^2}\,S^{\frac{1}{4}}(Z_2)\,dZ_2,$$

and so $$U_2(s) = (4/\pi)(X/S'(X))(I-(\surd 2+1)J), \tag{48}$$

where $$I+iJ = \int_{R_+} \frac{Z_1}{Z_1^2-X^2} S^{\frac{1}{4}}(Z_1)\,dZ_1. \tag{49}$$

By taking R_+ as a contour from 0 to ∞ passing above the pole at $Z_1 = X$, and removing the singular part of the integrand, we find that

$$I+iJ = \int_0^\infty \frac{X_1}{(X_1^2-X^2)^2}\{S^{\frac{1}{4}}(X_1)-S^{\frac{1}{4}}(X)-\tfrac{1}{4}S^{-\frac{3}{4}}(X)\,S'(X)\,(X_1-X)\}\,dX_1 \\ -\tfrac{1}{2}X^{-2}S^{\frac{1}{4}}(X)+\tfrac{1}{8}X^{-1}S^{-\frac{3}{4}}(X)\,S'(X)\,(1-\tfrac{1}{2}\pi i). \tag{50}$$

The contribution of J to $U_2(s)$ is therefore

$$\tfrac{1}{4}(\surd 2+1)\,S^{-\frac{3}{4}}(X) = -U_1(s),$$

and so

$$V_t(s) = \frac{4X}{\pi S'(X)}\left\{\int_0^\infty \frac{X_1}{(X_1^2-X^2)^2}\,(S^{\frac{1}{4}}(X_1)-S^{\frac{1}{4}}(X)-\tfrac{1}{4}S^{-\frac{3}{4}}(X)\,S'(X)\,(X_1-X))\,dX_1 \right. \\ \left. -\tfrac{1}{2}X^{-2}S^{\frac{1}{4}}(X)+\tfrac{1}{8}X^{-1}S^{-\frac{3}{4}}(X)\,S'(X)\right\}. \tag{51}$$

3. Second-order boundary layer: series solution

We now have to solve (22), (23), (24) with u_1, v_1 given by Glauert's solution and $p_1 = 0$, subject to the conditions (25) and (26) with $V_t(s)$ given by the analysis of §2. The continuity equation (24) is satisfied when the velocity components u_2, v_2 are given in terms of the second-order stream function ψ_2 by

$$u_2 = \partial\psi_2/\partial n, \quad v_2+\kappa n v_1 = -\partial\psi_2/\partial s. \tag{52}$$

After solving for p_2 we can write (22) in terms of the stream function as

$$\psi_{2nnn}-\psi_{1n}\psi_{2ns}-\psi_{2n}\psi_{1ns}+\psi_{1s}\psi_{2nn}+\psi_{2s}\psi_{1nn} = -H(s,n), \tag{53}$$

where
$$\begin{aligned} H(s,n) &= \kappa(s)\,(\psi_{1nn}+n\psi_{1nnn}+\psi_{1n}\psi_{1s})+\frac{\partial}{\partial s}\left(\kappa(s)\int_n^\infty \psi_{1n}^2\,dn\right) \\ &= \kappa(s)\,n\psi_{1nnn}+\kappa'(s)\int_n^\infty \psi_{1n}^2\,dn \\ &= \tfrac{1}{16}s^{-\frac{5}{4}}\{\kappa\eta f'''(\eta)+8s\kappa' g'^2(\eta)\}. \end{aligned} \tag{54}$$

We first consider the variation in Glauert's integral due to second-order effects. Since $u_2 \to V_t(s)$ as $n \to \infty$, we must modify the integral and write

$$\int_0^\infty \psi u(u-R^{-\frac{1}{2}}V_t(s)+O(R^{-1}))\,dn \\ = \int_0^\infty \psi_1 u_1^2\,dn + R^{-\frac{1}{2}}\int_0^\infty (\psi_2 u_1^2+2\psi_1 u_1 u_2-\psi_1 u_1 V_t)\,dn+O(R^{-1}). \tag{55}$$

Here $$\int_0^\infty \psi_1 u_1^2\,dn = \tfrac{1}{40}, \quad \int_0^\infty \psi_1 u_1\,dn = \tfrac{1}{2}s^{\frac{1}{2}}. \tag{56}$$

The equation corresponding to (6) is

$$\frac{\partial}{\partial s}(\psi_2 \psi_{1n}^2 + 2\psi_1 \psi_{1n} \psi_{2n}) - \frac{\partial}{\partial n}(\psi_2 \psi_{1nn} + \psi_1 \psi_{2nn} - \psi_{1n}\psi_{2n} + \psi_2 \psi_{1s}\psi_{1n} + \psi_1 \psi_{2s}\psi_{1n} + \psi_1 \psi_{1s}\psi_{2n}) = \psi_1 H(s,n). \tag{57}$$

Hence

$$\begin{aligned}\frac{d}{ds}\int_0^\infty (\psi_2 u_1^2 + 2\psi_1 u_1 u_2)\,dn &= \int_0^\infty \psi_1 H\,dn + \lim_{n\to\infty}(\psi_1 \psi_{1s}\psi_{2n}) \\ &= \int_0^\infty \tfrac{1}{4}s^{-\frac{1}{4}}(\kappa\eta f f''' + 2s\kappa' f'^2)\,d\eta + \tfrac{1}{4}s^{-\frac{1}{2}}V_t(s) \\ &= s^{-\frac{1}{4}}(\tfrac{1}{12}\kappa(s) + \tfrac{1}{9}s\kappa'(s)) + \tfrac{1}{4}s^{-\frac{1}{2}}V_t(s),\end{aligned} \tag{58}$$

which integrates to give

$$\int_0^\infty (\psi_2 u_1^2 + 2\psi_1 u_1 u_2)\,dn = \tfrac{1}{9}s^{\frac{3}{4}}\kappa(s) + \frac{1}{4}\int s^{-\frac{1}{2}}V_t(s)\,ds. \tag{59}$$

Since $$V_t(s) = -\tfrac{1}{4}(\sqrt{2}+1)\,s^{-\frac{3}{4}} + \sum_0^\infty d_n s^n \quad \text{for small } s,$$

$$\int s^{-\frac{1}{2}}V_t(s)\,ds = (\sqrt{2}+1)\,s^{-\frac{1}{4}} + \sum_0^\infty \frac{d_n}{n+\frac{1}{2}}s^{n+\frac{1}{2}} + \text{constant}, \tag{60}$$

and it is convenient to assume that the constant of (60) is absorbed into the first-order term by redefinition of the invariant F. Thus in terms of the physical variables

$$\begin{aligned}F^{-1}&\int_0^\infty \overline{\psi}\overline{u}\{\overline{u} - \overline{V}_t(\overline{s}) + O(R^{-1}U_c)\}\,d\overline{n} \\ &= 1 + 10R^{-\frac{1}{2}}\left\{\tfrac{4}{9}s^{\frac{3}{4}}\kappa(s) + \tfrac{3}{2}(\sqrt{2}+1)\,s^{-\frac{1}{4}} - 2\int_0^s s^{\frac{1}{2}}U_2'(s)\,ds\right\} + O(R^{-1}).\end{aligned} \tag{61}$$

When η is used as independent variable in place of n and the first-order solution (21) is inserted, (53) becomes

$$\frac{\partial^3\psi_2}{\partial\eta^3} + f\frac{\partial^2\psi_2}{\partial\eta^2} + 5f'\frac{\partial\psi_2}{\partial\eta} - 4s\left(f'\frac{\partial^2\psi_2}{\partial s\,\partial\eta} - f''\frac{\partial\psi_2}{\partial s}\right) = -4s(\kappa\eta f''' + 8s\kappa' g'^2), \tag{62}$$

with the boundary conditions

$$\psi_2 = \partial\psi_2/\partial\eta = 0 \quad \text{at} \quad \eta = 0, \quad \partial\psi_2/\partial\eta \to 4s^{\frac{3}{4}}V_t(s) \quad \text{as} \quad \eta\to\infty. \tag{63}$$

Following Van Dyke, the solution of (62) can be broken up into a displacement effect and a curvature effect. We write

$$\psi_2(s,\eta) = \psi_d(s,\eta) + \psi_c(s,\eta) = \sum_m c_m s^{m+1}\chi_m(\eta) + \sum_m \kappa_m s^{m+1}\phi_m(\eta), \tag{64}$$

where $$V_t(s) = \sum_m \tfrac{1}{4}c_m s^{m+\frac{1}{4}}, \tag{65}$$

$$\kappa(s) = \sum_m \kappa_m s^m. \tag{66}$$

In the series solution the cases of prime importance for ψ_d are

$$\left.\begin{aligned} m &= -1, \quad c_m = -(\sqrt{2}+1), \\ \text{and} \qquad m &= n-\tfrac{1}{4}, \quad c_m = 4d_n \quad (n = 0,1,2,\ldots), \end{aligned}\right\} \tag{67}$$

and for ψ_c they are $m = 0,1,2,\ldots$, but it will be of use for the integral solution to discuss the functions χ_m, ϕ_m for general values of m. The equations to be solved are

$$\chi_m''' + f\chi_m'' - (4m-1)f'\chi_m' + 4(m+1)f''\chi_m = 0, \tag{68}$$

$$\phi_m''' + f\phi_m'' - (4m-1)f'\phi_m' + 4(m+1)f''\phi_m = -4(\eta f''' + 8mg'^2), \tag{69}$$

with the boundary conditions

$$\chi_m(0) = \chi_m'(0) = 0, \quad \chi_m'(\infty) = 1, \tag{70}$$

$$\phi_m(0) = \phi_m'(0) = \phi_m'(\infty) = 0. \tag{71}$$

Equation (68) arises in Riley's (1962) study of the decay of perturbations to a wall jet in which he showed how it can be reduced to a hypergeometric equation. One solution of (68) is $f'(\eta)$, and the transformations

$$\left.\begin{aligned} \chi_m(\eta) &= f'(\eta)\,\sigma(\eta), \\ \sigma'(\eta) &= (1-\zeta)^{-1}\,\Omega(\zeta), \quad \zeta = g^3, \end{aligned}\right\} \tag{72}$$

lead to

$$\zeta(1-\zeta)\,\Omega'' + (\tfrac{5}{3} - \tfrac{8}{3}\zeta)\,\Omega' - \tfrac{2}{3}(4m+3)\,\Omega = 0, \tag{73}$$

which is a hypergeometric equation in which the parameters are given by

$$a+b = \tfrac{5}{3}, \quad ab = \tfrac{8}{3}m + 2, \quad c = \tfrac{5}{3}. \tag{74}$$

Equation (73) has solutions $\Omega = F(\zeta)$, $G(\zeta)$ where

$$\begin{aligned} F(\zeta) &= F(a,b;1;1-\zeta) \\ &= \frac{\Gamma(-\frac{2}{3})}{\Gamma(1-a)\,\Gamma(1-b)}\,F(a,b;\tfrac{5}{3};\zeta) + \frac{\Gamma(\frac{2}{3})}{\Gamma(a)\,\Gamma(b)}\,\zeta^{-\frac{2}{3}}\,F(1-a,1-b;\tfrac{1}{3};\zeta), \end{aligned} \tag{75}$$

$$G(\zeta) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(\frac{5}{3})}\,F(a,b;\tfrac{5}{3};\zeta) = -\sum_0^\infty \{\log(1-\zeta) + C_n\}\,\frac{(a)_n(b)_n}{(n!)^2}\,(1-\zeta)^n, \tag{76}$$

$$C_n = \psi(n+a) + \psi(n+b) - 2\psi(n+1). \tag{77}$$

The general solution of (68) is therefore

$$\chi_m = \zeta^{\frac{1}{3}}(1-\zeta)\left[A + B\int_{\epsilon_1}^{\zeta} \zeta^{-\frac{4}{3}}(1-\zeta)^{-2}F(\zeta)\,d\zeta + C\int_{\epsilon_2}^{\zeta} \zeta^{-\frac{4}{3}}(1-\zeta)^{-2}G(\zeta)\,d\zeta\right]. \tag{78}$$

The boundary conditions in terms of ζ are

$$\left.\begin{aligned} \chi_m \to 0, \quad \zeta^{\frac{2}{3}}\,d\chi_m/d\zeta \to 0 \quad &\text{as} \quad \zeta \to 0, \\ (1-\zeta)\,d\chi_m/d\zeta \to 1 \quad &\text{as} \quad \zeta \to 1. \end{aligned}\right\} \tag{79}$$

Hence $C = 1$, $B = 0$ and if we choose $\epsilon_2 = 0$ then $A = 0$. Thus

$$\chi_m(\eta) = g(1-g^3)\int_0^{g^3} \zeta^{-\frac{4}{3}}(1-\zeta)^{-2}G(\zeta)\,d\zeta. \tag{80}$$

This solution breaks down if a or b is a negative integer and these cases provide the eigenvalues of Riley's decaying perturbations. The first of these corresponds to a small change in Glauert's F and this is excluded by taking the constant in (60) as zero. The higher-order perturbations are all more singular at $s = 0$ than the first term $m = -1$ of (67). These are associated with the method of production of the wall jet, and it is assumed that they have all decayed over a length scale small compared with the characteristic length l of the surface.

A particular integral for ϕ_m can be found in the form

$$P(\eta) = \alpha + \beta(g^3 + \tfrac{1}{2}hf') + \lambda(\eta f - \tfrac{3}{2}\eta^2 f'), \tag{81}$$

where

$$\begin{aligned} h(\eta) &= \int_0^\eta g(\eta)\,d\eta \\ &= \eta - 2\sqrt{3}\tan^{-1}(g\sqrt{3}/(g+2)). \end{aligned} \tag{82}$$

On substituting in (69) and equating coefficients we find that

$$\left.\begin{aligned} \alpha &= -2(16m^3 + 42m^2 + 17m - 3)/\{(4m+7)(m+1)(2m+1)\}, \\ \beta &= 16m(2m+3)/\{(4m+7)(2m+1)\}, \quad \lambda = 4/(4m+7). \end{aligned}\right\} \tag{83}$$

The complementary function $Q(\eta)$ must therefore satisfy

$$Q(0) = -\alpha, \quad Q'(0) = 0, \quad Q'(\infty) = -\lambda. \tag{84}$$

$Q(\eta)$ is determined by the same method as χ_m but a modification is needed since $Q(0) \neq 0$. We thus obtain

$$\begin{aligned} \phi_m(\eta) = \alpha\left[g^3 + \tfrac{1}{3}g(1-g^3)\int_0^{g^3}\left\{\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(\frac{2}{3})}\zeta^{-\frac{4}{3}}(1-\zeta)^{-2}F(\zeta) - \zeta^{-\frac{4}{3}}\right\}d\zeta\right] + \beta(g^3 + \tfrac{1}{2}hf') \\ + \lambda\left[\eta f - \tfrac{3}{2}\eta^2 f' - g(1-g^3)\int_0^{g^3}\zeta^{-\frac{4}{3}}(1-\zeta)^{-2}G(\zeta)\,d\zeta\right]. \end{aligned} \tag{85}$$

The second-order skin friction is

$$\tau_2 = \left(\frac{\partial u_2}{\partial n}\right)_{n=0} = \tfrac{1}{16}s^{-\frac{3}{2}}\left[\sum_m c_m \chi_m''(0)\,s^{m+1} + \sum_m \kappa_m \phi_m''(0)\,s^{m+1}\right], \tag{86}$$

and from (80) and (85) we find

$$\chi_m''(0) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(\frac{2}{3})}, \tag{87}$$

$$\phi_m''(0) = -\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(\frac{2}{3})}\left(\frac{\Gamma(\frac{4}{3})}{\Gamma(1-a)\,\Gamma(1-b)}\alpha + \lambda\right). \tag{88}$$

Similarly as $n \to \infty$

$$\psi_2 - V_t(s)\,n \to \sum_m c_m \lim_{\eta\to\infty}(\chi_m - \eta)\,s^{m+1} + \sum_m \kappa_m \phi_m(\infty)\,s^{m+1}, \tag{89}$$

where

$$\chi_m - \eta \to -\left(\tfrac{3}{2}\log 3 + \frac{\pi}{2\sqrt{3}} + C_0 + 1\right), \tag{90}$$

$$\phi_m(\infty) = \left(1 + \frac{\Gamma(a)\,\Gamma(b)}{3\Gamma(\frac{2}{3})}\right)\alpha + \beta + \left(\tfrac{3}{2}\log 3 + \frac{\pi}{2\sqrt{3}} + C_0 + 1\right)\lambda. \tag{91}$$

We have seen that the case $m = -1$ corresponds to the displacement flow when the wall jet is on a plane surface. Equation (68) is then a second-order equation for χ'_{-1} and can be solved explicitly. It is easy to verify that

$$\chi_{-1}(\eta) = h - 5(f + \eta f'), \tag{92}$$

and so

$$\left.\begin{aligned} &\chi''_{-1}(0) = -3, \\ &\chi_{-1}(\eta) - \eta \to -(5 + \pi/\surd 3) \quad \text{as} \quad \eta \to \infty. \end{aligned}\right\} \tag{93}$$

The values in (93) agree with (87) and (90) since $a = 2, b = -\frac{1}{3}$. This case has also been treated by Plotkin (1970), and the analytical solution of Hayasi (1970) agrees with (92).

The case $m = -\frac{1}{2}$ is also of interest. Here $a = 1, b = \frac{2}{3}$ and

$$\chi_{-\frac{1}{2}}(\eta) = h^2 f' + h(4g^3 - 3) - f + 5\eta f', \tag{94}$$

$$\left.\begin{aligned} &\chi''_{-\frac{1}{2}}(0) = 1, \\ &\chi_{-\frac{1}{2}}(\eta) - \eta \to -(1 + \pi/\surd 3) \quad \text{as} \quad \eta \to \infty. \end{aligned}\right\} \tag{95}$$

The constants α, β, λ occurring in the expression for ϕ_m are singular for $m = -\frac{1}{2}$, -1 and $-\frac{7}{4}$ but of these only $-\frac{7}{4}$ is an eigenvalue. In fact ϕ_m is not singular at $m = -\frac{1}{2}$ and -1, and has only a simple pole at $m = -\frac{7}{4}$.

Although $m = -\frac{3}{4}$ is an eigenvalue the function $\phi_m(\eta)$ is not singular here. The particular integral

$$P(\eta) = 16g^3 + (8h - 2\eta^2)f' \tag{96}$$

satisfies all the boundary conditions, as does the complementary function

$$Q(\eta) = f + \eta f'. \tag{97}$$

Consequently

$$\phi_{-\frac{3}{4}}(\eta) = P(\eta) + kQ(\eta) \tag{98}$$

is a possible solution for any value of k. We can find the appropriate value by considering the limit as $b \to 0$ of $\phi''_m(0)$ or $\phi_m(\infty)$, since

$$P''(0) = 0, \quad Q''(0) = \tfrac{2}{3}, \quad P(\infty) = 16, \quad Q(\infty) = 1. \tag{99}$$

The result is

$$k = \tfrac{3}{2}\log 3 + \frac{\pi}{2\surd 3} - \frac{217}{18} = -9{\cdot}50. \tag{100}$$

In this case the radius of curvature is proportional to the boundary-layer thickness and so this is an example of what Van Dyke calls a 'jointly self-similar' solution. This case was studied by Wygnanski & Champagne (1968) on the basis of a set of boundary-layer equations that agree with Van Dyke's to second order, but differ at the third order. Wygnanski & Champagne looked for a similarity solution and obtained a generalization of Glauert's equation (9). It might seem that their solution should agree to second order with that given above, but they imposed the same boundary conditions $f(0) = f'(0) = 0$, $f(\infty) = 1$ as Glauert, so that they would have $k = -15$. An attempt was made to expand the solution of their equation, leaving $f(\infty)$ arbitrary, in powers of the curvature parameter, but this gave indeterminacy at the first power and impossibility at the second power. A further examination of Wygnanski & Champagne's equation is made in the appendix, where it is shown that it has no solution with

$$f(0) = f'(0) = f'(\infty) = 0$$

that tends to Glauert's as the curvature tends to 0.

A more satisfactory treatment of this problem was given by Lindow & Greber (1968). They also used equations that differ from Van Dyke's at the third order but realized that the similarity variable may differ from Glauert's. Their second-order solution agrees with (98) but they chose arbitrarily to take $k = -43/4$.

m	$\chi_m''(0)$	$\lim\,(\eta-\chi_m)$	m	$\phi_m''(0)$	$\phi_m(\infty)$
-1	-3	6·8138	0	$-0·73596$	3·7580
$-0·25$	$+3·1590\times10^{-1}$	4·3588	1	$+3·5746$	2·4490
$+0·75$	$2·2744\times10^{-2}$	5·9141	2	6·7380	1·8392
1·75	$3·8887\times10^{-3}$	6·5012	3	9·3788	1·4835
2·75	$9·2835\times10^{-4}$	6·8688	4	11·713	1·2490
3·75	$2·6883\times10^{-4}$	7·1371	5	13·842	1·0820
4·75	$8·8681\times10^{-5}$	7·3483	6	15·821	0·9566
5·75	$3·2193\times10^{-5}$	7·5227	7	17·684	0·8587
6·75	$1·2587\times10^{-5}$	7·6711	8	19·454	0·7800
7·75	$5·2246\times10^{-6}$	7·8003	9	21·146	0·7152
8·75	$2·2787\times10^{-6}$	7·9147	10	22·774	0·6610
9·75	$1·0364\times10^{-6}$	8·0174	11	24·345	0·6148
			12	25·866	0·5750

TABLE 1

Although analytical solutions have been given for the functions $\chi_m(\eta)$ and $\phi_m(\eta)$, these are inconvenient for numerical calculation. Solutions were computed for us by Dr Ian Gladwell directly from (68) and (69) for $\chi_m(\eta)$ with $m = -1, -\frac{1}{4}\,(1)\,\frac{23}{4}$ and for $\phi_m(\eta)$ with $m = 0\,(1)\,6\,(2)\,12$. Tables and graphs of these functions and their derivatives are given in the first author's M.Sc. thesis (Clark 1970). The values of $\chi_m''(0)$, $\lim\,(\eta-\chi_m)$, $\phi_m''(0)$ and $\phi_m(\infty)$ thus obtained agree closely with those given in table 1, which were derived from the analytical expressions (87), (90), (88) and (91).

4. Second-order boundary layer: integral solution

The series solution (64) is useful only when s is small, but we can find a general solution of (62) by means of the Mellin transformation. Since the solution for

$$V_t = U_1(s) = -\tfrac{1}{4}(\sqrt{2}+1)\,s^{-\frac{3}{4}}$$

is already known we can suppose that in (63) $V_t(s)$ is replaced by $U_2(s)$, so that from now on the suffix d refers to the additional displacement flow due to the curvature of the surface.

The Mellin transform of the stream function ψ_2 is

$$\psi^*(\tau,\eta) = \int_0^\infty \psi_2(s,\eta)\,s^{\tau-1}\,ds, \tag{101}$$

and the same transformation applied to equation (62) gives

$$\frac{\partial^3\psi^*}{\partial\eta^3}+f\frac{\partial^2\psi^*}{\partial\eta^2}+(4\tau+5)f'\frac{\partial\psi^*}{\partial\eta}-4\tau f''\psi^* = -4\kappa^*(\tau)\,(\eta f'''-8(\tau+1)\,g'^2), \quad (102)$$

where

$$\kappa^*(\tau) = \int_0^\infty s^\tau \kappa(s)\,ds \quad (103)$$

is the Mellin transform of $s\kappa(s)$. The boundary conditions for (102) are

$$\psi^* = \partial\psi^*/\partial\eta = 0 \quad \text{at} \quad \eta = 0, \quad \partial\psi^*/\partial\eta \to U^*(\tau) \quad \text{as} \quad \eta\to\infty, \quad (104)$$

where

$$U^*(\tau) = 4\int_0^\infty s^{\tau-\frac{1}{4}}\,U_2(s)\,ds. \quad (105)$$

The solution of (102), subject to the boundary conditions (104), is

$$\psi^*(\tau,\eta) = U^*(\tau)\,\chi_{-\tau-1}(\eta)+\kappa^*(\tau)\,\phi_{-\tau-1}(\eta). \quad (106)$$

The transformation inverse to (101) is

$$\psi_2(s,\eta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \psi^*(\tau,\eta)\,s^{-\tau}\,d\tau, \quad (107)$$

where c is chosen so that the integral converges. Consequently the displacement and curvature effects are given respectively by

$$\psi_d(s,\eta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} U^*(\tau)\,\chi_{-\tau-1}(\eta)\,s^{-\tau}\,d\tau, \quad (108)$$

$$\psi_c(s,\eta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \kappa^*(\tau)\,\phi_{-\tau-1}(\eta)\,s^{-\tau}\,d\tau. \quad (109)$$

The results of §3 show that $\chi_{-\tau-1}(\eta)$ is a regular function of τ except for simple poles at

$$\tau = \tfrac{3}{8}n^2+\tfrac{5}{8}n-\tfrac{1}{4} \quad (n = 0, 1, 2, \ldots), \quad (110)$$

and that $\phi_{-\tau-1}(\eta)$ behaves similarly except that $n = 0$ gives a regular point. The contour of integration must pass to the left of all these poles. Since the functions $\chi_{-\tau-1}(\eta)$ and $\phi_{-\tau-1}(\eta)$ are complicated, attention will be directed to the contributions to the second-order skin friction, namely

$$\tau_{d,c}(s) = \tfrac{1}{16}s^{-\frac{3}{4}}(\partial^2\psi_{d,c}/\partial\eta^2)_{\eta=0}, \quad (111)$$

and to the outer limit function

$$L = \lim_{n\to\infty}\,(\psi_2-U_2(s)\,n) = L_d+L_c. \quad (112)$$

The simplest case is

$$\tau_d(s) = \tfrac{1}{16}s^{-\frac{3}{4}}\cdot\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} U^*(\tau)\,\chi''_{-\tau-1}(0)\,s^{-\tau}\,d\tau, \quad (113)$$

where

$$\chi''_{-\tau-1}(0) = \Gamma(a)\,\Gamma(b)/\Gamma(\tfrac{2}{3}) \quad (114)$$

and

$$a, b = \tfrac{1}{6}(5\pm(49+96\tau)^{\frac{1}{2}}). \quad (115)$$

As $|\tau|\to\infty$, except in a sector $|\arg\tau|<\epsilon$, $\chi''_{-\tau-1}(0)$ is exponentially small. Consequently we can substitute the integral (105) for $U^*(\tau)$ into (113) and obtain, on reversing the order of integration,

$$\tau_d(s) = \tfrac{1}{4}s^{-\frac{3}{2}}\int_0^\infty r^{-\frac{1}{4}}U_2(r)\left\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\chi''_{-\tau-1}(0)\left(\frac{r}{s}\right)^\tau d\tau\right\}dr. \tag{116}$$

For $r>s$ the inner integral can be evaluated by means of a large semicircle to the left and shown to be zero. Hence

$$\tau_d = \tfrac{1}{4}s^{-\frac{3}{2}}\int_0^1\left\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\chi''_{-\tau-1}(0)\,t^\tau d\tau\right\}(st)^{-\frac{1}{4}}U_2(st)\,s\,dt$$

$$= s^{-\frac{3}{4}}\int_0^1 f_d(t)\,U_2(st)\,dt, \tag{117}$$

where
$$f_d(t) = \frac{t^{-\frac{1}{4}}}{8\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(\frac{2}{3})}\,t^\tau d\tau. \tag{118}$$

In order to evaluate $f_d(t)$ it is convenient to deform the contour of integration into a loop from infinity round the poles of the integrand, which are on the positive real axis. The substitution

$$\tau = \tfrac{3}{8}(\theta^2-49/36) \tag{119}$$

then leads to

$$f_d(t) = \frac{t^{-\frac{1}{4}}}{8\pi i}\int_C\frac{\Gamma(\frac{5}{6}+\theta)\,\Gamma(\frac{5}{6}-\theta)}{\Gamma(\frac{2}{3})}\,t^{\frac{3}{8}(\theta^2-49/36)}.\tfrac{3}{4}\theta\,d\theta, \tag{120}$$

where C is a path from $\infty e^{i(\pi-\delta)}$ to $\infty e^{i\delta}$ in the upper half plane, and $0<\delta<\frac{1}{4}\pi$. Since the integrand is an odd function of θ the integral is $(-2\pi i)$ times the sum of the residues at the poles $\theta=\frac{5}{6}+n$ for $n=0,1,2,\ldots$. Thus

$$f_d(t) = \frac{1}{8}\sum_0^\infty(-1)^n\frac{(\frac{5}{3})_n}{n!}(n+\tfrac{5}{6})\,t^{\frac{3}{8}n^2+\frac{5}{8}n-\frac{1}{2}}. \tag{121}$$

The series (121) converges for $|t|<1$ and rapidly unless $1-|t|$ is quite small. In order to estimate the function $f_d(t)$ when $t\to1$ we put

$$t=e^{-h},\quad \theta=\zeta/h \tag{122}$$

and write (120) as

$$f_d(t) = \frac{3}{32i}\frac{\exp(73h/96)}{\Gamma(\frac{2}{3})\,h^2}\int_C\frac{\Gamma(\frac{5}{6}+\zeta/h)\exp(-3\zeta^2/8h)}{\Gamma(\frac{1}{6}+\zeta/h)\sin(\frac{5}{6}\pi-\pi\zeta/h)}\,\zeta\,d\zeta. \tag{123}$$

For $h\to0$ we can substitute the asymptotic forms for the Γ functions and write

$$2i\sin(\tfrac{5}{6}\pi-\pi\zeta/h)\sim\exp(\tfrac{5}{6}\pi i-\pi i\zeta/h).$$

This gives
$$f_d(t)\sim\frac{3}{16}\frac{h^{-\frac{4}{3}}}{\Gamma(\frac{2}{3})}\int_C\zeta^{\frac{5}{3}}\exp\{-\tfrac{5}{6}\pi i+h^{-1}(\pi i\zeta-\tfrac{3}{8}\zeta^2)\}\,d\zeta,$$

and the integral can be estimated by the saddle-point method, whence

$$f_d(t)\sim\frac{3}{16}\frac{\sqrt{2}}{\Gamma(\frac{2}{3})}\left(\frac{4\pi}{3(1-t)}\right)^{\frac{13}{6}}\exp\left(-\frac{2\pi^2}{3(1-t)}\right). \tag{124}$$

The outer limit for the displacement term is

$$L_d(s) = \lim_{\eta\to\infty}\{\psi_d(s,\,\eta) - 4s^{\frac{3}{4}}U_2(s)\,\eta\}$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} U^*(\tau)\lim_{\eta\to\infty}\{\chi_{-\tau-1}(\eta)-\eta\}\,s^{-\tau}\,d\tau. \qquad (125)$$

From (90) and (77)

$$\eta - \chi_{-\tau-1}(\eta) \to \tfrac{3}{2}\log 3 + (\pi/2\sqrt{3}) + 1 + 2\gamma + \psi(a) + \psi(b),$$

and as $|\tau|\to\infty$, a and $b\to\infty$, so that (except in a sector $|\arg\tau| < \epsilon$)

$$\psi(a)+\psi(b) = \log a - \tfrac{1}{2}a^{-1} - \tfrac{1}{12}a^{-2} + \tfrac{1}{120}a^{-4} + O(a^{-6})$$
$$+\log b - \tfrac{1}{2}b^{-1} - \tfrac{1}{12}b^{-2} + \tfrac{1}{120}b^{-4} + O(b^{-6})$$
$$= \log(-\tfrac{8}{3}\tau) + \tfrac{1}{2}\tau^{-1} - \tfrac{119}{960}\tau^{-2} + O(\tau^{-3}), \qquad (126)$$

after use has been made of equations (74). Consequently we can write

$$\lim_{\eta\to\infty}\{\chi_{-\tau-1}(\eta)-\eta\} = -\log(-8\sqrt{3}\tau) - (\pi/2\sqrt{3}) - 1 - 2\gamma - \Delta(\tau), \qquad (127)$$

where $\Delta(\tau)\to 0$ as $\tau\to\infty$. The contribution of $\Delta(\tau)$ to $L_d(s)$ is, proceeding as with τ_d,

$$L_\Delta(s) = s^{\frac{3}{4}}\int_0^1 U_2(st)\,g_d(t)\,dt, \qquad (128)$$

where

$$g_d(t) = -\frac{2}{\pi i}t^{-\frac{1}{4}}\int_{c-i\infty}^{c+i\infty}\Delta(\tau)\,t^{\tau}\,d\tau$$
$$= -\frac{2}{\pi i}t^{-\frac{1}{4}}\int_{c-i\infty}^{c+i\infty}\{\psi(a)+\psi(b)-\log(-\tfrac{8}{3}\tau)\}\,t^{\tau}\,d\tau$$
$$= -\frac{2}{\pi i}t^{-\frac{1}{4}}\int_C\{\psi(\tfrac{5}{6}+\theta)+\psi(\tfrac{5}{6}-\theta)-\log(\tfrac{49}{36}-\theta^2)\}\,t^{\frac{3}{8}\theta^2-49/96}\,.\,\tfrac{3}{4}\theta\,d\theta$$
$$= \frac{2}{\pi i}\frac{t^{-\frac{1}{4}}}{\log t}\int_C\left\{\psi'(\tfrac{5}{6}+\theta)-\psi'(\tfrac{5}{6}-\theta)-\frac{2\theta}{\theta^2-\frac{49}{36}}\right\}t^{\frac{3}{8}\theta^2-49/96}\,d\theta.$$

The integral can be evaluated by considering the residues at $\theta = \frac{7}{6}$ and $n+\frac{5}{6}$ $(n = 0, 1, 2, \ldots)$, with the result that

$$g_d(t) = \frac{4t^{-\frac{1}{4}}}{\log t} + 3\sum_0^\infty(n+\tfrac{5}{6})\,t^{\frac{3}{8}n^2+\frac{5}{8}n-\frac{1}{4}}. \qquad (129)$$

It can also be shown that

$$-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}U^*(\tau)\log(-\tau)\,s^{-\tau}\,d\tau = 4s^{\frac{3}{4}}\int_0^1\frac{U_2(s)-t^{-\frac{1}{4}}U_2(st)}{\log t}\,dt, \qquad (130)$$

so that

$$L_d(s) = -(2\log 192 + (2\pi/\sqrt{3}) + 4 + 8\gamma)\,s^{\frac{3}{4}}U_2(s)$$
$$+ s^{\frac{3}{4}}\int_0^1\{U_2(st)\,g_d(t) + (4/\log t)\,(U_2(s)-t^{-\frac{1}{4}}U_2(st))\}\,dt. \qquad (131)$$

The asymptotic behaviour of $\Delta(\tau)$ as $\tau\to\infty$ is obtained from (126) and may be used to infer the form of $g_d(t)$ as $t\to 1$ by considering a large loop as the contour of integration. In this way it is found that

$$g_d(t) = 2 + (\tfrac{239}{240})(1-t) + O(1-t)^2, \qquad (132)$$

and so the integrand in (131) tends to $3U_2(s)-4sU_2'(s)$ as $t\to 1$.

The contribution of the curvature to the skin friction is

$$\tau_c(s) = \tfrac{1}{16}s^{-\frac{3}{4}}.\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\kappa^*(\tau)\,\phi''_{-\tau-1}(0)\,s^{-\tau}\,d\tau. \tag{133}$$

Since, as will be shown, $\phi''_{-\tau-1}(0) = O(\tau^{\frac{4}{3}})$ for large $|\tau|$ it is convenient to write this as

$$\tau_c(s) = \frac{s^{-\frac{3}{4}}}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tau\kappa^*(\tau)\,(\tfrac{1}{16}\tau^{-1}\phi''_{-\tau-1}(0))\,s^{-\tau}\,d\tau$$

$$= s^{-\frac{3}{4}}\int_0^1\kappa_1(st)f_c(t)\,dt, \tag{134}$$

where
$$\kappa_1(s) = d(s\kappa(s))/ds \tag{135}$$

and
$$f_c(t) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tfrac{1}{16}\tau^{-1}\phi''_{-\tau-1}(0)\,t^{\tau}\,d\tau. \tag{136}$$

After α has been expressed in partial fractions, the method used for $f_d(t)$ gives

$$f_c(t) = -\frac{17}{28}+\frac{5}{108}\left(3\log 3+\frac{\pi}{\sqrt{3}}+\frac{671}{81}\right)t^{\frac{4}{3}}$$
$$+\frac{1}{6}\sum_2^\infty\left\{\frac{(\frac{2}{3})_{n+1}}{n!}A_n+\frac{2(-1)^{n+1}}{(n-1)(n+\frac{8}{3})}\right\}\frac{(\frac{2}{3})_{n+1}(n+\frac{5}{6})}{n!(n-\frac{1}{3})(n+2)}t^{\frac{3}{8}n^2+\frac{5}{8}n-\frac{1}{4}}, \tag{137}$$

where
$$A_n = 1-\frac{\frac{9}{5}}{(n-1)(n+\frac{8}{3})}+\frac{\frac{8}{3}}{(n-\frac{1}{3})(n+2)}+\frac{\frac{4}{5}}{(n+\frac{2}{3})(n+1)}. \tag{138}$$

The behaviour of $f_c(t)$ as $t\to 1$ is again found by considering a large loop integral in the τ plane. In $\phi''_{-\tau-1}(0)$ the term multiplying λ in (88) is exponentially small so that except in $|\arg\tau| < \epsilon$

$$\phi''_{-\tau-1}(0) = -\frac{\Gamma(\frac{4}{3})}{\Gamma(\frac{2}{3})}\frac{\Gamma(a)}{\Gamma(a-\frac{2}{3})}\frac{\Gamma(b)}{\Gamma(b-\frac{2}{3})}\alpha+\text{exp small}$$

$$= 16\frac{\Gamma(\frac{4}{3})}{\Gamma(\frac{2}{3})}(-\tfrac{1}{3}\tau)^{\frac{4}{3}}(1+\tfrac{211}{216}\tau^{-1}+O(\tau^{-2})), \tag{139}$$

and hence
$$f_c(t) = \frac{(1-t)^{-\frac{7}{3}}}{3^{\frac{4}{3}}\Gamma(\frac{2}{3})}(1-\tfrac{235}{72}(1-t)+O(1-t)^2). \tag{140}$$

The outer limit of the curvature stream function is

$$L_c(s) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\kappa^*(\tau)\,\phi_{-\tau-1}(\infty)\,s^{-\tau}\,d\tau$$

$$= s\int_0^1\kappa_1(st)\,g_c(t)\,dt, \tag{141}$$

where
$$g_c(t) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tau^{-1}\phi_{-\tau-1}(\infty)\,t^{\tau}\,d\tau$$

$$= \frac{4}{7}\left(12\log 3-\frac{5\pi}{3\sqrt{3}}+5\right)-\frac{2}{3}\left(3\log 3+\frac{\pi}{\sqrt{3}}+\frac{671}{81}\right)t^{\frac{4}{3}}$$
$$+\frac{8}{3}\sum_2^\infty\left\{(-1)^n\frac{(\frac{2}{3})_{n+1}}{n!}A_n-\frac{2}{(n-1)(n+\frac{8}{3})}\right\}\frac{n+\frac{5}{6}}{(n-\frac{1}{3})(n+2)}t^{\frac{3}{8}n^2+\frac{5}{8}n-\frac{1}{4}}. \tag{142}$$

For large $|\tau|$, except in $|\arg \tau| < \epsilon$,

$$\phi_{-\tau-1}(\infty) \sim \alpha+\beta+(\tfrac{3}{2}\log 3+(\pi/2\sqrt{3})+1+2\gamma+\psi(a)+\psi(b))\lambda$$
$$\sim -(\log(-8\sqrt{3}\tau)+(\pi/2\sqrt{3})+2\gamma+\tfrac{1}{2})(\tau^{-1}-\tfrac{3}{4}\tau^{-2}+\ldots)+\tfrac{5}{2}\tau^{-2}+\ldots. \quad (143)$$

From this it follows that as $t \to 1$

$$g_c(t) \sim (\log(8\sqrt{3}/(1-t))+(\pi/2\sqrt{3})+\gamma+\tfrac{3}{2})(1-t+\tfrac{1}{8}(1-t)^2+\ldots)$$
$$+\tfrac{5}{16}(1-t)^2+\ldots. \quad (144)$$

Table 2 gives numerical values of the functions f_d, g_d, f_c and g_c for $0 \leqslant t \leqslant 1$.

t	f_d	g_d	f_c	g_c
0	—	—	−0·60714	8·6631
0·01	1·00350	22·8035	−0·58753	8·3807
0·02	0·68267	15·7378	−0·57416	8·1883
0·03	0·53555	12·6486	−0·56243	8·0196
0·04	0·44501	10·8274	−0·55164	7·8649
0·05	0·38137	9·5966	−0·54151	7·7197
0·1	0·21303	6·6038	−0·49647	7·0796
0·15	0·13189	5·3175	−0·45656	6·5228
0·2	0·08255	4·5663	−0·41934	6·0169
0·25	0·05038	4·0612	−0·38372	5·5478
0·3	0·02926	3·6926	−0·34900	5·1079
0·4	0·00774	3·1816	−0·28027	4·2958
0·5	0·00120	2·8373	−0·20933	3·5508
0·6	0·00007	2·5857	−0·13123	2·8498
0·7	0	2·3916	−0·03694	2·1724
0·75	0	2·3099	+0·02217	1·8367
0·8	0	2·2361	0·09626	1·4994
0·85	0	2·1691	0·19794	1·1573
0·9	0	2·1079	0·36083	0·8053
0·95	0	2·0518	0·72607	0·4340
0·96	0	2·0411	0·87706	0·3556
0·97	0	2·0306	1·10390	0·2749
0·98	0	2·0202	1·50026	0·1911
0·99	0	2·0100	2·46598	0·1023
1	0	2	—	0

TABLE 2

5. Applications: flow over a parabolic cylinder

The methods described in the previous sections have been applied to the cases in which the wall jet is placed at the vertex of a parabolic cylinder, both outside and inside. If the length l is taken as the radius of curvature at the vertex the surface has equation $y = \mp\frac{1}{2}x^2$ in non-dimensional form, with the flow in the region $y \pm \frac{1}{2}x^2 > 0$. Suffices o and i will be used to denote the cases of flow outside and inside the parabola.

The conformal transformation from the half plane $Y > 0$ to the outside of the parabola is

$$z = F(Z) = Z - \tfrac{1}{2}iZ^2, \quad (145)$$

so that $x = X$ on the parabola. Then

$$F'(Z) = 1-iZ, \quad \bar{F}'(Z) = 1+iZ, \tag{146}$$

and hence
$$S(Z) = S_o(Z) = \int_0^Z (1+Z^2)^{\frac{1}{2}}\,dZ$$
$$= \tfrac{1}{2}\{Z(1+Z^2)^{\frac{1}{2}} + \log(Z+(1+Z^2)^{\frac{1}{2}})\}, \tag{147}$$

where the square root is real and positive for Z real and positive. The function $S_o(Z)$ has the properties

$$S_o(Z) = Z + \tfrac{1}{6}Z^3 - \tfrac{1}{40}Z^5 + O(Z^7) \tag{148}$$

for $Z \to 0$, and for $Z \to \infty$

$$S_o(Z) = \tfrac{1}{2}Z^2 + \tfrac{1}{2}\log(2Z) + \tfrac{1}{4} + \tfrac{1}{16}Z^{-2} - \tfrac{1}{64}Z^{-4} + O(Z^{-6}). \tag{149}$$

Also $S_o(Z)$ is regular in the domain formed by cutting the Z plane from i to $i\infty$ and from $-i$ to $-i\infty$ and does not vanish except at $Z = 0$.

For the inside of the parabola we introduce an intermediate ζ plane where

$$Z = \sinh(\tfrac{1}{2}\pi\zeta), \tag{150}$$

$$z = \zeta + \tfrac{1}{2}i\zeta^2. \tag{151}$$

The strip $0 < \mathscr{I}\zeta < 1$ transforms into the upper half of the Z plane cut from $Z = i$ to $i\infty$, and into the inside of the parabola cut from $z = \frac{1}{2}i$ to $i\infty$. Although each of the transformations (150), (151) is singular at $\zeta = i$, the combined transformation from Z to z is not singular and the two sides of each cut join up. In this case we have

$$S(Z) = S_i(Z) = S_o(\zeta), \tag{152}$$

and it is convenient to integrate in the ζ plane. From (150), $X = \sinh(\frac{1}{2}\pi x)$ on the parabola.

The coefficients for the outer flow were computed by numerical integration along a contour composed of straight line segments joining ih, $(i+1)h$, h and $+\infty$. The integration program was written for us by Dr Ian Gladwell and was adapted for use in the later integrations.

Since $S_o(Z)$ is an odd function the flow on the outside of the parabola is given by

$$\frac{dw_2}{dZ} = \sum_0^\infty b_{o_{2n+1}} Z^{2n+1}, \tag{153}$$

where
$$b_{o_{2n+1}} = (2n+2)\mathscr{R}\left\{\frac{1+(1+\sqrt{2})\,i}{\pi}\int_{ih}^{\infty} \frac{S_o^{\frac{1}{2}}(Z)}{Z^{2n+3}}\,dZ\right\}. \tag{154}$$

The numerical integration was carried along the real axis until it was possible to estimate the remainder from (149). Dr Gladwell's computations gave

$$b_{o_1} = 0{\cdot}286819, \quad b_{o_3} = -0{\cdot}107848, \quad b_{o_5} = 0{\cdot}057011, \tag{155}$$

and consistent values were found for $h = 0{\cdot}5(0{\cdot}1)0{\cdot}9$. Since $s = S_o(X)$, we find from (148) that

$$U_{2o}(s) = \sum_0^\infty d_{o_{2n+1}} s^{2n+1}, \tag{156}$$

where
$$d_{o_1} = 0{\cdot}286819, \quad d_{o_3} = -0{\cdot}29906, \quad d_{o_5} = 0{\cdot}3752. \tag{157}$$

The corresponding coefficients for flow inside the parabola were found by using ζ as the variable of integration, so that

$$b_{i_{2n+1}} = (n+1)\mathscr{R}\left\{(1+(1+\surd 2)\,i)\int_{ih}^{\infty}\frac{S_o^{\frac{1}{2}}(\zeta)\cosh\left(\frac{1}{2}\pi\zeta\right)}{\sinh^{2n+3}\left(\frac{1}{2}\pi\zeta\right)}\,d\zeta\right\}. \quad (158)$$

The results were

$$b_{i_1} = -0{\cdot}141387, \quad b_{i_3} = 0{\cdot}086582, \quad b_{i_5} = -0{\cdot}064795, \quad (159)$$

and these gave the coefficients for $U_{2i}(s)$ as

$$d_{i_1} = -0{\cdot}348857, \quad d_{i_3} = 0{\cdot}18584, \quad d_{i_5} = -0{\cdot}1650. \quad (160)$$

The curvature of the surface (positive for flow over the outside of the parabola) is

$$\kappa(s) = (1+x^2)^{-\frac{3}{2}} = \sum_0^{\infty}\kappa_{2n}s^{2n}, \quad (161)$$

where

$$\left.\begin{aligned}&\kappa_0 = 1, \quad \kappa_2 = -1{\cdot}5, \quad \kappa_4 = 2{\cdot}375, \quad \kappa_6 = -3{\cdot}8042,\\ &\kappa_8 = 6{\cdot}1210, \quad \kappa_{10} = -9{\cdot}8717, \quad \kappa_{12} = 15{\cdot}821.\end{aligned}\right\} \quad (162)$$

In the following results the suffix f denotes the value for a flat surface, d the additional displacement contribution corresponding to $U_2(s)$, and c the contribution of the curvature terms for the outside of the parabola. The last of these changes sign when the inside of the parabola is considered.

For the second-order skin friction

$$\tau_f = \tfrac{3}{16}(\surd 2+1)\,s^{-\frac{3}{2}} = 0{\cdot}452665\,s^{-\frac{3}{2}}, \quad (163)$$

and from (157), (160), (162) and table 1

$$\left.\begin{aligned}\tau_{do} &= s^{\frac{1}{2}}(1{\cdot}631\times 10^{-3} - 6{\cdot}94\times 10^{-5}s^2 + 8{\cdot}3\times 10^{-6}s^4 + O(s^6)),\\ \tau_{di} &= s^{\frac{1}{2}}(-1{\cdot}984\times 10^{-3} + 4{\cdot}31\times 10^{-5}s^2 - 3{\cdot}7\times 10^{-6}s^4 + O(s^6)),\\ \tau_c &= s^{-\frac{1}{2}}(-0{\cdot}04600 - 0{\cdot}63169\,s^2 + 1{\cdot}7387\,s^4 - 3{\cdot}762\,s^6\\ &\qquad + 7{\cdot}442\,s^8 - 14{\cdot}05\,s^{10} + 25{\cdot}6\,s^{12} + O(s^{14})).\end{aligned}\right\} \quad (164)$$

The outer limit function $L = \lim\,(\psi_2 - U_2(s)\,n)$ gives

$$\left.\begin{aligned}L_f &= (\surd 2+1)\,(5+\pi/\surd 3) = 16{\cdot}4500,\\ L_{do} &= s^{\frac{3}{2}}(-6{\cdot}785 + 8{\cdot}22\,s^2 - 11{\cdot}0\,s^4 + O(s^6)),\\ L_{di} &= s^{\frac{3}{2}}(8{\cdot}253 - 5{\cdot}11\,s^2 + 4{\cdot}85\,s^4 + O(s^6)),\\ L_c &= 3{\cdot}7580\,s - 2{\cdot}759\,s^3 + 2{\cdot}966\,s^5 - 3{\cdot}64\,s^7\\ &\qquad + 4{\cdot}8\,s^9 - 6{\cdot}5\,s^{11} + 9{\cdot}1\,s^{13} + O(s^{15}).\end{aligned}\right\} \quad (165)$$

These series are satisfactory for $|s| < 0{\cdot}5$, except for L_{do} and L_{di}, and the values thus obtained provide a check on those computed by the integral method of §4.

The speed of the outer flow is given by (51), but for computational convenience the range of integration was divided into three parts and a partial integration carried out over the middle range to give

$$V_t = \frac{4}{\pi X S'(X)}\left\{\int_0^a \frac{tS^{\frac{1}{2}}(Xt)}{(t^2-1)^2}\,dt + \tfrac{1}{8}X\int_a^b \frac{S^{-\frac{1}{2}}(Xt)\,S'(Xt) - S^{-\frac{1}{2}}(X)\,S'(X)}{t^2-1}\,dt\right.$$
$$\left. + \int_b^{\infty}\frac{tS^{\frac{1}{2}}(Xt)}{(t^2-1)^2}\,dt - \frac{1}{2}\frac{S^{\frac{1}{2}}(Xa)}{1-a^2} - \frac{1}{2}\frac{S^{\frac{1}{2}}(Xb)}{b^2-1}\right\} + \frac{1}{4\pi}\log\left(\frac{1+a}{1-a}\,\frac{b-1}{b+1}\right)S^{-\frac{1}{2}}(X), \quad (166)$$

where $X_1 = Xt$, $0 < a < 1 < b$, $s = S(X)$. For the outside flow the first two integrals were evaluated numerically and the third estimated from the asymptotic behaviour (149) of $S_o(X)$. In the case of the inside flow the further transformation

$$\left.\begin{aligned} X &= \sinh(\tfrac{1}{2}\pi\xi), & Xt &= \sinh(\tfrac{1}{2}\pi\xi\theta), \\ Xa &= \sinh(\tfrac{1}{2}\pi\xi\alpha), & Xb &= \sinh(\tfrac{1}{2}\pi\xi\beta), \end{aligned}\right\} \tag{167}$$

was made and β chosen large enough for the third integral, which is exponentially small in β, to be neglected. The results were checked by comparison with the series for small s and with the asymptotic behaviour for $s \to \infty$ which will now be described.

When $X \to \infty$ in (51) the range of integration may be divided into $(0, aX^{\frac{1}{2}})$ and $(aX^{\frac{1}{2}}, \infty)$. In the former we may expand $(X_1^2 - X^2)^{-2}$ in powers of X_1/X and in the latter use the asymptotic form for $S^{\frac{1}{4}}(X_1)$ as well as for $S^{\frac{1}{4}}(X)$. In this way it was found, using (149), that for the outside flow

$$\begin{aligned} V_{t_o}(s) = -\tfrac{1}{4}s^{-\frac{3}{4}}\left(1 + \frac{3\pi}{8s} + \frac{21\pi^2}{128s^2}\right) + c_1 s^{-2} \\ + (\tfrac{1}{2}c_1 \log(8s) + \tfrac{1}{4}c_1 + c_2)\, s^{-3} + O(s^{-\frac{15}{4}} \log s), \end{aligned} \tag{168}$$

where

$$c_1 = \frac{1}{\pi}\int_0^\infty X\{S_o^{\frac{1}{4}}(X) - 2^{\frac{1}{4}}X^{\frac{1}{2}}[1 + \tfrac{1}{4}X^{-2}(\log(2X) + \tfrac{1}{2})]\}\, dX,$$

$$\begin{aligned} c_2 = \frac{1}{\pi}\int_0^\infty X^3\{S_o^{\frac{1}{4}}(X) - 2^{\frac{1}{4}}X^{\frac{1}{2}}[1 + \tfrac{1}{4}X^{-2}(\log(2X) + \tfrac{1}{2}) \\ + \tfrac{1}{32}X^{-4}(-3(\log(2X) + \tfrac{1}{2})^2 + 1)]\}\, dX. \end{aligned}$$

Numerical integration gave $c_1 = 0{\cdot}09413$, $c_2 = 0{\cdot}2864$. When the inside of the parabola is considered, the contribution of the range $(0, aX^{\frac{1}{2}})$ is exponentially small in terms of s and the remainder gives

$$V_{ti}(s) \sim -\frac{1}{\sqrt{2}}s^{-\frac{1}{4}}\left(1 + \frac{\log(8s) - \frac{2}{3}}{8s} + \frac{\frac{3}{2}\log^2(8s) - \frac{20}{3}\log(8s) + \frac{23}{6}}{(8s)^2} + \ldots\right). \tag{169}$$

The leading term of (168) corresponds to taking the suction velocity $\frac{1}{4}s^{-\frac{3}{4}}$ to act on the axis $x = 0$, $y < 0$; the leading term of (169) to dividing the total inflow $2s^{\frac{1}{4}}$ by the channel width $2x$.

After the programme for computing $U_2(s)$ had been tested it was used to provide values of the integrands for τ_d and L_d. The integrands for τ_c and L_c involve

$$\kappa_1(s) = (1 + x^2)^{-\frac{3}{2}} - 3sx(1 + x^2)^{-3}. \tag{170}$$

Where necessary the ranges of integration were divided in order to deal separately with the singularities at $t = 0$ and $t = 1$. Values of $U_2'(s)$, needed for the integrand for L_d at $t = 1$, were obtained by numerical differentiation. The variation of Glauert's integral as given by (61) was also computed from

$$\left.\begin{aligned} F_f &= 15(\sqrt{2} + 1)s^{-\frac{1}{4}}, \\ F_d &= 10\int_0^s s^{-\frac{1}{2}}U_2(s)\, ds - 20s^{\frac{1}{2}}U_2(s), \\ F_c &= \tfrac{40}{9}s^{\frac{3}{4}}\kappa(s). \end{aligned}\right\} \tag{171}$$

A few of the results of these computations are given in tables 3, 4, 5 and 6.

	Flat	Outside parabola			Inside parabola	
s	U_1	U_{2o}	V_{to}	κ	U_{2i}	V_{ti}
0	$-0{\cdot}604s^{-\frac{3}{4}}$	$0{\cdot}287s$	$-0{\cdot}604s^{-\frac{3}{4}}$	1	$-0{\cdot}349s$	$-0{\cdot}604s^{-\frac{3}{4}}$
0·1	−3·394	0·0284	−3·366	0·9852	−0·0347	−3·429
0·2	−2·018	0·0551	−1·963	0·9436	−0·0683	−2·086
0·5	−1·015	0·1148	−0·9002	0·7310	−0·1553	−1·170
1	−0·6036	0·1501	−0·4535	0·4152	−0·2454	−0·8490
2	−0·3589	0·1399	−0·2190	0·1642	−0·3111	−0·6699
5	−0·1805	0·0908	−0·0897	0·0385	−0·3279	−0·5084
10	−0·1073	0·0544	−0·0490	0·0127	−0·3090	−0·4163
20	−0·0638	0·0360	−0·0278	0·00427	−0·2799	−0·3437
50	−0·0321	0·0185	−0·0136	0·00104	−0·2374	−0·2695
100	−0·0191	0·0111	−0·0080	0·00036	−0·2062	−0·2253
∞	$-0{\cdot}604s^{-\frac{3}{4}}$	$0{\cdot}354s^{-\frac{3}{4}}$	$-0{\cdot}25s^{-\frac{3}{4}}$	$0{\cdot}354s^{-\frac{3}{2}}$	$-0{\cdot}707s^{-\frac{1}{4}}$	$-0{\cdot}707s^{-\frac{1}{4}}$

TABLE 3

	Flat	Outside parabola			Inside parabola	
s	τ_f	τ_{do}	τ_c	τ_o	τ_{di}	τ_i
0	$0{\cdot}453s^{-\frac{3}{2}}$	$0{\cdot}00163s^{\frac{1}{4}}$	$-0{\cdot}0460s^{-\frac{1}{2}}$	$0{\cdot}453s^{-\frac{3}{2}}$	$-0{\cdot}00198s^{\frac{1}{4}}$	$0{\cdot}453s^{-\frac{3}{2}}$
0·1	14·32	0·00092	−0·1649	14·16	−0·00112	14·48
0·2	5·061	0·00108	−0·1536	4·918	−0·00133	5·213
0·5	1·280	0·00136	−0·1900	1·091	−0·00166	1·468
1	0·4527	0·00157	−0·1652	0·2891	−0·00194	0·6160
2	0·1600	0·00170	−0·0797	0·0820	−0·00220	0·2375
5	0·0405	0·00155	−0·0180	0·0230	−0·00228	0·0562
10	0·0143	0·00119	−0·00507	0·0104	−0·00202	0·0173
20	0·00506	0·00078	−0·00136	0·00448	−0·00158	0·00484
50	0·00128	0·00038	−0·00023	0·00143	−0·00098	+0·00053
100	0·00045	0·00019	−0·00006	0·00058	−0·00062	−0·00011
∞	$0{\cdot}453s^{-\frac{3}{2}}$	$0{\cdot}143s^{-\frac{5}{4}}$	$-0{\cdot}698s^{-2}$	$0{\cdot}143s^{-\frac{5}{4}}$	$-0{\cdot}177s^{-1}$	$-0{\cdot}177s^{-1}$

TABLE 4

	Flat	Outside parabola			Inside parabola	
s	L_f	L_{do}	L_c	L_o	L_{di}	L_i
0	16·45	$-6{\cdot}78s^{\frac{7}{4}}$	$3{\cdot}76s$	16·45	$8{\cdot}25s^{\frac{7}{4}}$	16·45
0·1	16·45	−0·1192	0·3731	16·70	0·1458	16·22
0·2	16·45	−0·3872	0·7304	16·79	0·4818	16·20
0·5	16·45	−1·558	1·605	16·50	2·145	16·99
1	16·45	−3·201	2·366	15·62	5·509	19·59
2	16·45	−4·434	2·701	14·72	11·08	24·83
5	16·45	−4·494	2·424	14·38	21·19	35·22
10	16·45	−3·745	1·990	14·69	31·65	46·31
20	16·45	−2·617	1·551	15·38	45·68	60·58
50	16·45	−0·699	1·066	16·82	72·36	87·84
100	16·45	+1·073	0·786	17·31	101·3	117·0
∞	16·45	$+3{\cdot}43s^{\frac{1}{4}}$	$9{\cdot}30s^{-\frac{1}{2}}$	$3{\cdot}43s^{\frac{1}{4}}$	$7{\cdot}96s^{\frac{1}{2}}$	$7{\cdot}96s^{\frac{1}{2}}$

TABLE 5

	Flat	Outside parabola			Inside parabola	
s	F_f	F_{do}	F_c	F_o	F_{di}	F_i
0	$36{\cdot}2s^{-\frac{1}{4}}$	$-3{\cdot}82s^{\frac{3}{2}}$	$4{\cdot}44s^{\frac{3}{4}}$	$36{\cdot}2s^{-\frac{1}{4}}$	$4{\cdot}65s^{\frac{3}{2}}$	$36{\cdot}2s^{-\frac{1}{4}}$
0·1	64·40	−0·1193	0·7787	65·06	0·1461	63·76
0·2	54·15	−0·3246	1·254	55·08	0·4050	53·30
0·5	43·06	−1·011	1·932	43·99	1·415	42·55
1	36·21	−1·590	1·845	36·47	2·939	37·31
2	30·45	−1·314	1·228	30·36	4·476	33·70
5	24·22	+0·457	0·5726	25·25	4·987	28·63
10	20·36	2·175	0·3174	22·86	3·952	24·00
20	17·12	3·842	0·1794	21·15	+1·740	18·68
50	13·62	5·780	0·0868	19·49	−3·082	10·45
100	11·45	7·017	0·0508	18·52	−8·341	+3·06
∞	$36{\cdot}2s^{-\frac{1}{4}}$	13·70	$1{\cdot}57s^{-\frac{3}{4}}$	13·70	$-14{\cdot}1s^{\frac{1}{4}}$	$-14{\cdot}1s^{\frac{1}{4}}$

TABLE 6

6. Conclusion

The integral solution requires a suitable path of integration to exist for the integrals (108), (109). The function $\chi_{-\tau-1}(\eta)$ is regular in $\mathscr{R}\tau < -\frac{1}{4}$, and $\phi_{-\tau-1}(\eta)$ is regular in $\mathscr{R}\tau < \frac{3}{4}$. The integral (105) for $U^*(\tau)$ converges at $s = 0$ provided $\mathscr{R}\tau > -\frac{3}{4}$ in general, and $\mathscr{R}\tau > -\frac{7}{4}$ in the symmetrical case; the integral (103) for $\kappa^*(\tau)$ converges for $\mathscr{R}\tau > -1$ if the curvature is finite at $s = 0$. For the case of a parabolic surface the integral for $U^*(\tau)$ converges as $s \to \infty$ provided $\mathscr{R}\tau < 0$ in the case of external flow, and provided $\mathscr{R}\tau < -\frac{1}{2}$ for internal flow; that for $\kappa^*(\tau)$ converges if $\mathscr{R}\tau < \frac{1}{2}$. Thus in each case there is a strip of the τ plane within which the path of integration may be drawn. The series solution for small s may be recovered by considering the residues at the poles of $U^*(\tau)$ and $\kappa^*(\tau)$ to the left of the path; the asymptotic behaviour for large s depends on the singularities of the integrands to the right.

For the outside flow the first singularity to the right is that of $\chi_{-\tau-1}(\eta)$ at $\tau = -\frac{1}{4}$, so that as $s \to \infty$

$$\psi_{do} \sim -U_o^*(-\tfrac{1}{4})s^{\frac{1}{4}} \lim_{\tau\to-\frac{1}{4}} \{(\tau+\tfrac{1}{4})\chi_{-\tau-1}(\eta)\}$$
$$\sim \frac{5}{2}\int_0^\infty r^{-\frac{1}{4}}U_{2o}(r)\,dr\,.\,s^{\frac{1}{4}}(f+\eta f') \sim 3{\cdot}43s^{\frac{1}{4}}(f+\eta f'). \qquad (172)$$

This result follows also from considering the variation of Glauert's integral when $s \to \infty$ and represents an effective change of origin of the wall jet due to the perturbation by the outer flow. For the inside flow the strip in the τ plane is bounded to the right by the requirement of convergence of $U^*(\tau)$ as $s \to \infty$ and hence

$$\psi_{di} \sim -\chi_{-\frac{1}{2}}(\eta)\,s^{\frac{1}{2}} \lim_{\tau\to-\frac{1}{2}} \{(\tau+\tfrac{1}{2})\,U_i^*(\tau)\}$$
$$\sim -(8s)^{\frac{1}{2}}\,(h^2f'+h(4g^3-3)-f+5\eta f'), \qquad (173)$$

from (169) and (94). The integral for ψ_c converges in $-1 < \mathscr{R}\tau < \frac{1}{2}$ since the first singularity of $\phi_{-\tau-1}(\eta)$ is at $\tau = \frac{3}{4}$. Thus

$$\psi_c \sim (8s)^{-\frac{1}{2}}\phi_{-\frac{3}{2}}(\eta). \tag{174}$$

It follows from these results that as $s \to \infty$, $\psi_d \gg \psi_f \gg \psi_c$ for both external and internal flow, and the series solution for small s makes $\psi_f \gg \psi_c \gg \psi_d$ in symmetrical flow and $\psi_f \gg \psi_d \gg \psi_c$ in asymmetrical flow, as $s \to 0$. The results proved for $s \to \infty$ are special to the case of a parabolic surface, though the inequalities are probably of wider application. The main conclusion to be drawn is that for the wall jet the effect of the curvature terms in (22) to (24) is less important than that of the displacement flow appearing in the boundary condition (26).

Finally, it should be noted that for flow inside the parabola the solution must fail when $s = O(R^2)$, since then the boundary-layer thickness becomes comparable with the channel width and the velocity of the second-order outer flow is comparable with that in the first-order boundary layer. The second-order skin friction is negative for $s > 80$ and the flow may separate when $s = O(R^2)$, but the present method of analysis does not seem able to decide this point since it requires Glauert's solution to be a valid first approximation. These difficulties do not arise in the case of flow outside the parabola.†

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Appendix

Wygnanski & Champagne (1968) based their study of the wall jet on a curved surface on the equations

$$u\frac{\partial u}{\partial s} + v\frac{\partial}{\partial n}\{(1+\kappa n)\,u\} = -\frac{\partial p}{\partial s} + \frac{\partial}{\partial n}\left\{(1+\kappa n)\frac{\partial u}{\partial n}\right\}, \tag{A 1}$$

$$\kappa u^2 = \frac{\partial p}{\partial n}, \quad \frac{\partial u}{\partial s} + \frac{\partial}{\partial n}\{(1+\kappa n)\,v\} = 0. \qquad \text{(A 2), (A 3)}$$

If these equations are expanded in powers of $R^{-\frac{1}{2}}$ they agree to terms $O(R^{-\frac{1}{2}})$ with Van Dyke's first- and second-order boundary-layer equations, but differ in

† *Note added in proof*: After this paper had been accepted, a referee sent us copies of a paper to be published by Plotkin (1971), which treats a wall jet on the outside of a parabola by numerical integration of the second-order boundary layer equations. Plotkin's calculations were confined to the region $0 < s \leqslant 1{\cdot}5$, and his results for the effect of curvature on the skin friction agree well with ours. The displacement effect is not comparable, since Plotkin considered a one-sided wall jet with flow in the positive direction only. He did not investigate the resultant boundary layer in the region $s < 0$, but this appears not to influence the flow in $s > 0$ to $O(R^{-\frac{1}{2}})$.

terms $O(R^{-1})$. Wygnanski & Champagne sought a similarity solution of equations (A 1) to (A 3) in the form

$$\psi = s^{\frac{1}{4}} f(\eta), \quad \eta = \tfrac{1}{4} s^{-\frac{3}{4}} n, \quad \kappa = \tfrac{1}{4} k s^{\frac{3}{4}}, \tag{A 4}$$

and obtained, assuming that $u \to 0$ as $n \to \infty$,

$$f''' + ff'' + 2f'^2 = -k\left[\eta f''' + f'' + \frac{ff' + 3k\eta^2 f'^2}{1+k\eta} - 4\int_\eta^\infty f'^2\, d\eta\right], \tag{A 5}$$

with the boundary conditions

$$f(0) = f'(0) = f'(\infty) = 0. \tag{A 6}$$

For $k = 0$, (A 5) reduces to Glauert's equation (9). Following Glauert, multiply (A 5) by f and integrate from 0 to η. Then

$$ff'' - \tfrac{1}{2}f'^2 + f^2 f' = -k\left[\eta(ff'' - \tfrac{1}{2}f'^2) + \int_0^\eta \left\{\tfrac{1}{2}f'^2 + \frac{f^2 f' + 3k\eta^2 ff'^2}{1+k\eta} - 4f\int_\eta^\infty f'^2\, d\eta\right\} d\eta\right]. \tag{A 7}$$

Since $f'(\infty) = 0$, either $k = 0$ or

$$\int_0^\infty \left\{\tfrac{1}{2}f'^2 + \frac{f^2 f' + 3k\eta^2 ff'^2}{1+k\eta} - 4f\int_\eta^\infty f'^2\, d\eta\right\} d\eta = 0. \tag{A 8}$$

This condition is satisfied when $k = 0$ and f is Glauert's function since then

$$\int_\eta^\infty f'^2\, d\eta = f'' + ff'$$

and

$$\int_0^\infty \{\tfrac{1}{2}f'^2 + f^2 f' - 4f(f'' + ff')\}\, d\eta = \int_0^\infty \{-ff'' - f'^2\}\, d\eta = 0.$$

For general values of k we obtain similarly from (A 5)

$$\int_\eta^\infty f'^2\, d\eta = f'' + ff' + k\left\{\eta f'' - \int_\eta^\infty \left(\frac{ff' + 3k\eta^2 f'^2}{1+k\eta} - 4\int_\eta^\infty f'^2\, d\eta\right) d\eta\right\},$$

and (A 8) becomes, using (A 7) to eliminate $f^2 f'$,

$$\int_0^\infty (-ff'' - f'^2 + kF)\, d\eta = 0,$$

where

$$F = \frac{3\eta^2 ff'^2 - \eta f^2 f'}{1+k\eta} - \eta(ff'' + \tfrac{3}{2}f'^2) + 4f\int_\eta^\infty \left(\frac{ff' + 3k\eta^2 f'^2}{1+k\eta} - 4\int_\eta^\infty f'^2\, d\eta\right) d\eta$$
$$+ 3\int_0^\eta \left(\tfrac{1}{2}f'^2 + \frac{f^2 f' + 3k\eta^2 ff'^2}{1+k\eta} - 4f\int_\eta^\infty f'^2\, d\eta\right) d\eta. \tag{A 9}$$

Hence if $f(\eta)$ satisfies (A 5) and (A 6) with $k \neq 0$

$$\int_0^\infty F\, d\eta = 0. \tag{A 10}$$

Now if as $k \to 0, f(\eta) \to f_0(\eta)$ where $f_0(\eta)$ is Glauert's function, then $F \to F_0$ where

$$F_0 = 3\eta^2 f_0 f_0'^2 - \eta(f_0^2 f_0' + f_0 f_0'' + \tfrac{3}{2} f_0'^2) + 4f_0 \int_\eta^\infty \left(f_0 f_0' - 4\int_\eta^\infty f_0'^2 \, d\eta\right) d\eta$$
$$+ 3\int_0^\eta \left(\tfrac{1}{2} f_0'^2 + f_0^2 f_0' - 4 f_0 \int_\eta^\infty f_0'^2 \, d\eta\right) d\eta. \quad \text{(A 11)}$$

After some calculation,

$$\int_0^\infty F_0 \, d\eta = \frac{3}{5}\sum_1^\infty \frac{(\frac{2}{3})_n}{n!\,n^2} + \frac{(\pi + 3\sqrt{3}\log 3)^2}{40} + \pi\sqrt{3} + 9\log 3 - \tfrac{683}{40} > 0{\cdot}6. \quad \text{(A 12)}$$

Consequently (A 5) has no solution, subject to the conditions (A 6), that tends to Glauert's function as $k \to 0$.

The third-order boundary-layer equations are

$$\psi_{1n}\psi_{3ns} + \psi_{2n}\psi_{2ns} + \psi_{3n}\psi_{1ns} - \psi_{1s}(\psi_{3nn} + \kappa\psi_{2n} - \kappa^2 n\psi_{1n})$$
$$- \psi_{2s}(\psi_{2nn} + \kappa\psi_{1n}) - \psi_{3s}\psi_{1nn}$$
$$= -p_{3s} + \psi_{3nnn} + \kappa n\psi_{2nnn} + \kappa\psi_{2nn} - \kappa^2\psi_{1n} + \psi_{1ssn}, \quad \text{(A 13)}$$

$$-\psi_{1n}\psi_{1ss} - 2\kappa\psi_{1n}\psi_{2n} + \psi_{1s}\psi_{1ns} = -p_{3n} - \kappa n p_{2n} - \psi_{1nns}, \quad \text{(A 14)}$$

where

$$u_3 = \psi_{3n}, \quad v_3 + \kappa n v_2 = -\psi_{3s}. \quad \text{(A 15)}$$

The forcing terms in these equations may be classified into (i) products of second-order terms, (ii) (curvature) × (first-order) × (second-order), (iii) (curvature)^2 × (first-order), (iv) first-order terms.

The fourth type is omitted when Murphy's (1953) equations for the boundary layer on a curved surface are expanded in powers of $R^{-\frac{1}{2}}$ but all the others are included. We may also divide the solution of equations (A 13) to (A 15) into various contributions, one of which comprises the terms quadratic in the curvature, and we may seek a solution for this quadratic curvature effect in the joint-similarity case by writing

$$\kappa = \tfrac{1}{4} k s^{-\frac{3}{4}}, \quad \psi_{3cc} = k^2 s^{\frac{1}{4}} \omega(\eta). \quad \text{(A 16)}$$

It may be shown that if $\omega(0) = \omega'(0) = 0$, then

$$\omega'(\infty) = 11 - 6\log 3 - 2\pi/\sqrt{3} = 0{\cdot}781. \quad \text{(A 17)}$$

This result explains why it is inappropriate to look for a similarity solution of equations (A 1) to (A 3), or those of Murphy, in the form (A 4) with $u \to 0$ as $n \to \infty$.

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